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On the Optimal Value Function of a Linearly Perturbed Quadratic Program

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Abstract. The optimal value function $(c, b) \mapsto \varphi(c, b)$ of the quadratic program $\min\{\frac{1}{2}x^T Dx + c^T x : Ax \geqslant b\}$, where $D \in R_S^{n\times n}$ is a given symmetric matrix, $A \in R^{m\times n}$ a given matrix, $c \in R^n$ and $b \in R^m$ are the linear perturbations, is considered. It is proved that φ is directionally differentiable at any point $\overline{w} = (\overline{c}, \overline{b})$ in its effective domain $W := \{w = (c, b) \in R^n \times R^m : -\infty < \varphi(c, b) < +\infty\}$. Formulae for computing the directional derivative $\varphi'(\overline{w}; z)$ of φ at $\overline{w}$ in a direction $z = (u, v) \in R^n \times R^m$ are obtained. We also present an example showing that, in general, φ is not piecewise linear-quadratic on W. The preceding (unpublished) example of Klatte is also discussed.

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1. Introduction

Consider the quadratic programming (QP, for brevity) problem with linear constraints

$$\begin{aligned} &\text{minimize} \quad f(x, c) := \tfrac{1}{2}x^T Dx + c^T x \\ &\text{subject to} \quad x \in \Delta(A, b) := \{x \in R^n : Ax \geqslant b\}, \end{aligned} \tag{1.1}$$

where $D \in R_S^{n\times n}$ and $A \in R^{m\times n}$ are given matrices, $c \in R^n$ and $b \in R^m$ are given vectors. Here $R_S^{n\times n}$ denotes the set of $(n \times n)$-symmetric matrices, and the apex ^{T} stands for the transposition. Denote by $S(D, A, c, b)$, $\mathrm{Sol}(D, A, c, b)$, $\mathrm{loc}(D, A, c, b)$ and $\varphi(D, A, c, d)$, respectively, the set of the Karush–Kuhn–Tucker points, the set of the (global) solutions, the set of the local solutions, and the optimal value of (1.1). Thus, in particular, $\varphi(D, A, c, b) = \inf\{f(x, c) : x \in \Delta(A, b)\}$. By convention, $\varphi(D, A, c, b) = +\infty$ if $\Delta(A, b) = \emptyset$.

Suppose that the matrices D and A are not subject to perturbation. We are interested in studying the function $\varphi(D, A, \cdot, \cdot) : R^n \times R^m \to \bar{R}$, $(c, b) \mapsto \varphi(D, A, c, b)$. Here (c, b) represents the pair of linear perturbations in problem (1.1). Klatte [8] has proved that $\varphi(D, A, \cdot, \cdot)$ is Lipschitz on every bounded subset of its effective domain

$$W := \{(c, b) \in R^n \times R^m : -\infty < \varphi(D, A, c, b) < +\infty\}. \tag{1.2}$$

The aim of this paper is to obtain some results related to the directional differentiability and the piecewise linear-quadratic property of the optimal value function $\varphi(D, A, \cdot, \cdot)$. By abuse of notation, we shall write $\varphi(c, b)$ instead of $\varphi(D, A, c, b)$.

It will be shown that although φ is not a convex function, it enjoys the important property of convex functions of being directionally differentiable at any point in its effective domain. Formulae for computing the directional derivative $\varphi'(\bar{w}; z)$ of φ at an arbitrary point $\bar{w} = (\bar{c}, \bar{b}) \in W$ in a direction $z = (u, v) \in R^n \times R^m$ are established.

Differential property of the optimal value function in quadratic programming has been addressed, for example, in Refs. [1, 3, 7, 13]. Continuity of the optimal value function $(D, A, c, b) \mapsto \varphi(D, A, c, b)$ has been characterized by Tam [14].

The notion of *piecewise linear-quadratic function* (plq function, for brevity) was introduced by Rockafellar [10]. The class of plq functions has been investigated systematically in Rockafellar and Wets [11]. In particular, the topics like subdifferential calculation, dualization, and optimization involving plq functions, are studied in the book. It is known that if D is positive semidefinite, i.e., $x^T D x \geqslant 0$ for all $x \in R^n$, then $\varphi(c, b)$ is piecewise linear-quadratic on W which, in this case, is a polyhedral convex cone. In the case where D is not a positive semidefinite matrix, the closed cone W may be nonconvex; but it can be represented as the union of finitely many polyhedral convex cones ([8], Theorem 2). It is of interest to know *whether* $\varphi(c, b)$ *is still a plq function on* W. This question was raised by one of the two anonymous referees of Tam [14]. We will construct an example which shows that, in general, φ is not a plq function on W. This example exposes well the structure of the class of optimal value functions under our consideration.

We give some auxiliary results in Section 2 and establish the directional differentiability of the function $\varphi(c, b)$ in Section 3. We study the piecewise linear-quadratic property of φ in Section 4.

2. Auxiliary Results

Fix a pair $(D, A) \in R_S^{n \times n} \times R^{m \times n}$ and consider problem (1.1), where $(c, b) \in R^n \times R^m$ is the pair of linear perturbations. Following [8], we consider the auxiliary problem

$$\text{minimize} \quad \tfrac{1}{2}(c^T x + b^T \lambda) \quad \text{subject to} \quad (x, \lambda) \in P_{\mathrm{KKT}}(c, b), \tag{2.1}$$

where

$$\begin{aligned} P_{\mathrm{KKT}}(c, b) = \{(x, \lambda) \in R^n \times R^m : Dx - A^T \lambda + c = 0, \\ Ax \geqslant b, \quad \lambda \geqslant 0, \ \lambda^T (Ax - b) = 0\}. \end{aligned} \tag{2.2}$$

Elements of $P_{\mathrm{KKT}}(c, b)$ are called the Karush–Kuhn–Tucker pairs of (1.1). Let

$$\varphi_{\mathrm{KKT}}(c, b) = \inf\left\{\tfrac{1}{2}(c^T x + b^T \lambda) : (x, \lambda) \in P_{\mathrm{KKT}}(c, b)\right\} \tag{2.3}$$

be the optimal value of problem (2.1). As in the preceding section, we set

$$\varphi(c, b) = \inf\left\{\tfrac{1}{2} x^T Dx + c^T x : Ax \geqslant b, \ x \in R^n\right\}. \tag{2.4}$$

Denote by $\mathrm{Sol}(c, b)$ and $\mathrm{Sol}_{\mathrm{KKT}}(c, b)$ the solution sets of (1.1) and of (2.1), respectively.

LEMMA 2.1 (see [8], p. 820). *If* $\mathrm{Sol}(c, b)$ *is nonempty then* $\mathrm{Sol}_{\mathrm{KKT}}(c, b)$ *is nonempty, and*

$$\begin{aligned} \mathrm{Sol}(c, b) &= \pi_{R^n}(\mathrm{Sol}_{\mathrm{KKT}}(c, b)), \\ \varphi(c, b) &= \varphi_{\mathrm{KKT}}(c, b), \end{aligned}$$

where, by definition, $\pi_{R^n}(x, \lambda) = x$ *for every* $(x, \lambda) \in R^n \times R^m$.

Note that the set W defined by (1.2) coincides with the effective domain of the multifunction $\mathrm{Sol}(D, A, \cdot, \cdot)$, that is

$$\begin{aligned} W &= \{(c, b) \in R^n \times R^m : -\infty < \varphi(c, b) < +\infty\} \\ &= \{(c, b) \in R^n \times R^m : \mathrm{Sol}(c, b) \neq \emptyset\}. \end{aligned} \tag{2.5}$$

Indeed, for any pair $(c, b) \in R^n \times R^m$, if $\mathrm{Sol}(c, b) \neq \emptyset$ then $-\infty < \varphi(c, b) < +\infty$. Conversely, if $-\infty < \varphi(c, b) < +\infty$ then $\Delta(A, b)$ is nonempty and the function $f(\cdot, c)$ is bounded below on $\Delta(A, b)$. By the Frank–Wolfe theorem (see, for instance, [4]), (1.1) must have a solution, i.e., $\mathrm{Sol}(c, b) \neq \emptyset$.

Taking account of (2.5), we can formulate the results from [8] concerning the optimal value function $\varphi(c, b)$ as follows.

LEMMA 2.2 ([8], Theorem 2). *The effective domain* W *of* φ *is the union of a finitely many polyhedral convex cones, i.e., there exists a finite number of polyhedral convex cones* $W_i \subset R^n \times R^m$ $(i = 1, 2, \ldots, s)$ *such that*

$$W = \bigcup_{i=1}^{s} W_i. \tag{2.6}$$

LEMMA 2.3 ([8], Theorem 3). *The function φ is Lipschitz on every bounded subset $\Omega \subset W$, i.e., for each bounded subset $\Omega \subset W$ there exists a constant $k_\Omega > 0$ such that*

$$\|\varphi(c', b') - \varphi(c, b)\| \leqslant k_\Omega(\|c' - c\| + \|b' - b\|)$$

for any (c, b), $(c', b') \in \Omega$.

For each subset $I \subset \{1, 2, \ldots, m\}$, we define

$$\begin{aligned} P^I_{\mathrm{KKT}}(c, b) = \{(x, \lambda) \in R^n \times R^m : {} & Dx - A^T\lambda + c = 0, \\ & A_i x \geqslant b_i,\ \lambda_i = 0\ (\forall i \in I), \\ & A_j x = b_j,\ \lambda_j \geqslant 0\ (\forall j \notin I)\} \end{aligned}$$

where A_i ($i \in \{1, \ldots, m\}$) denotes the i-th row of the matrix A and b_i is the i-th component of b. It is clear that

$$P_{\mathrm{KKT}}(c, b) = \bigcup_{I \subset \{1, \ldots, m\}} P^I_{\mathrm{KKT}}(c, b). \tag{2.7}$$

Note that $P^I_{\mathrm{KKT}}(c, b)$ is the solution set of the following system of linear equalities and inequalities:

$$\begin{aligned} Dx - A^T\lambda + c &= 0, \\ A_I x \geqslant b_I, \quad \lambda_I &= 0 \\ A_J x = b_J, \quad \lambda_J &\geqslant 0, \\ x \in R^n, \quad \lambda &\in R^m, \end{aligned} \tag{2.8}$$

where $J = \{1, 2, \ldots, m\} \setminus I$ and, as usual, A_J denotes the matrix composed by the rows A_j ($j \in J$) of A, and λ_I is the vector with the components λ_i ($i \in I$). Define

$$\varphi^I_{\mathrm{KKT}}(c, b) = \inf\left\{\tfrac{1}{2}(c^T x + b^T\lambda) : (x, \lambda) \in P^I_{\mathrm{KKT}}(c, b)\right\}. \tag{2.9}$$

Thus $\varphi^I_{\mathrm{KKT}}(c, b)$ is the optimal value of the *linear programming problem* whose objective function is $\frac{1}{2}(c^T x + b^T\lambda)$ and whose constraints are described by (2.8). It turns out that, for any $I \subset \{1, 2, \ldots, m\}$, the effective domain of $\varphi^I_{\mathrm{KKT}}(\cdot)$ is a polyhedral convex cone (see [9], p. 11) on which the function admits a linear-quadratic representation. Namely, using the concept of pseudo-inverse matrix one can establish the following result.

LEMMA 2.4 (see [2], Theorem 5.5.2). *The effective domain*

$$\mathrm{dom}\ \varphi^I_{\mathrm{KKT}} = \{(c, b) \in R^n \times R^m : -\infty < \varphi^I_{\mathrm{KKT}}(c, b) < +\infty\}$$

is a polyhedral convex cone and there exist a matrix $M_I \in R^{(n+m)\times(n+m)}$ and a vector $q_I \in R^{n+m}$ such that

$$\varphi^I_{\mathrm{KKT}}(c,b) = \frac{1}{2}\begin{pmatrix} c \\ b \end{pmatrix}^T M_I \begin{pmatrix} c \\ b \end{pmatrix} + q_I^T \begin{pmatrix} c \\ b \end{pmatrix}$$

for every $(c,b) \in \mathrm{dom}\ \varphi^I_{\mathrm{KKT}}$.

The following useful fact follows from Lemma 2.1.

LEMMA 2.5. *For any* $(c,b) \in W$, *it holds*

$$\varphi(c,b) = \min\{\varphi^I_{\mathrm{KKT}}(c,b) : I \subset \{1,2,\dots,m\}\}. \tag{2.10}$$

Proof. From (2.3), (2.7) and (2.9) we deduce that

$$\varphi_{\mathrm{KKT}}(c,b) = \min\{\varphi^I_{\mathrm{KKT}}(c,b) : I \subset \{1,2,\dots,m\}\}.$$

Combining this with the formula $\varphi(c,b) = \varphi_{\mathrm{KKT}}(c,b)$ we obtain (2.10). □

REMARK 2.1. From (2.10) it follows that, for any $I \subset \{1,2,\dots,m\}$ and for any $(c,b) \in W$, $\varphi^I_{\mathrm{KKT}}(c,b) > -\infty$.

REMARK 2.2. It may happen that for some pairs $(c,b) \in W$ the function φ^I_{KKT} has the value $+\infty$. Note that $\varphi^I_{\mathrm{KKT}}(c,b) = +\infty$ if and only if the solution set of (2.8) is empty. The example considered in Section 4 will illustrate this situation.

REMARK 2.3. If D is a positive semidefinite matrix then (1.1) is a convex QP problem and the equality $\varphi^{I_1}_{\mathrm{KKT}}(c,b) = \varphi^{I_2}_{\mathrm{KKT}}(c,b)$ holds for any index sets I_1, $I_2 \subset \{1,2,\dots,m\}$ and for any point $(c,b) \in \mathrm{dom}\ \varphi^{I_1}_{\mathrm{KKT}} \cap \mathrm{dom}\ \varphi^{I_2}_{\mathrm{KKT}}$. The last equality is valid because any KKT point of a convex QP problem is a global solution.

3. Directional Differentiability

Recall ([9], p. 13) that a subset $K \subset R^p$ is called a *cone* if $tx \in K$ whenever $x \in K$ and $t > 0$. (The origin itself may or may not be included in K.)

PROPOSITION 3.1. *Let* W *be defined by* (1.2), $L = \{b : \Delta(A,b) \neq \emptyset\}$,

$$Z_1 = \{(c,b) : \varphi(c,b) = +\infty\}, \quad Z_2 = \{(c,b) : \varphi(c,b) = -\infty\}.$$

Then Z_1 *is an open cone,* W *is a closed cone, and* Z_2 *is a cone which is relatively open in the polyhedral convex cone* $R^n \times L \subset R^n \times R^m$. *Moreover,*

$$R^n \times L = W \cup Z_2, \quad R^n \times R^m = W \cup Z_2 \cup Z_1,$$
$$Z_1 = (R^n \times R^m) \setminus (R^n \times L). \tag{3.1}$$

The easy proof of this proposition is omitted.

THEOREM 3.1. *The optimal value function φ defined in* (2.4) *is directionally differentiable on W, i.e., for any $\bar{w} = (\bar{c}, \bar{b}) \in W$ and for any $z = (u, v) \in R^n \times R^m$ there exists the directional derivative*

$$\varphi'(\bar{w}; z) := \lim_{t \downarrow 0} \frac{\varphi(\bar{w} + tz) - \varphi(\bar{w})}{t} \tag{3.2}$$

of φ at $\bar{w}$ in direction z.

Proof. Let $\bar{w} = (\bar{c}, \bar{b}) \in W$ and $z = (u, v) \in R^n \times R^m$ be given arbitrarily. If $z = 0$ then it is obvious that $\varphi'(\bar{w}; z) = 0$. Assume that $z \neq 0$. We first prove that one of the following three cases must occur:

(c1) there exists $\bar{t} > 0$ such that $\bar{w} + tz \in Z_1$ for every $t \in (0, \bar{t}\,]$,
(c2) there exists $\bar{t} > 0$ such that $\bar{w} + tz \in Z_2$ for every $t \in (0, \bar{t}\,]$,
(c3) there exists $\bar{t} > 0$ such that $\bar{w} + tz \in W$ for every $t \in (0, \bar{t}\,]$,

where the cones Z_1 and Z_2 have been defined in Proposition 3.1. For this purpose, suppose that (c3) fails to hold. We have to show that, in this case, (c1) or (c2) must occur. Since (c3) is not valid, we can find a decreasing sequence $t_k \to 0+$ such that $\bar{w} + t_k z \notin W$ for every $k \in N$, where N denotes the set of the positive integers. By (3.1), for each $k \in N$, we must have $\bar{w} + t_k z \in Z_1$ or $\bar{w} + t_k z \in Z_2$. Hence, there exists a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that

$$\bar{w} + t_{k_i} z \in Z_1 \quad (\forall i \in N), \tag{3.3}$$

or

$$\bar{w} + t_{k_i} z \in Z_2 \quad (\forall i \in N). \tag{3.4}$$

Consider the case where (3.3) is fulfilled. If there exists an $\hat{t} \in (0, t_{k_1})$ such that $\bar{w} + \hat{t}z \in R^n \times L$ then, by the convexity of the set $R^n \times L$,

$$\{\bar{w} + tz : t \in [0, \hat{t}\,]\} \subset R^n \times L.$$

By virtue of the first equality in (3.1), this yields $\varphi(\bar{w} + tz) \neq +\infty$ for every $t \in [0, \hat{t}\,]$, contradicting (3.3). Thus (3.3) implies that $\bar{w} + tz \notin R^n \times L$ for every $t \in (0, t_{k_1})$. Then, the third equality in (3.1) shows that $\bar{w} + tz \in Z_1$ for every $t \in (0, t_{k_1})$. Putting $\bar{t} = t_{k_1}$ we see that (c1) holds.

Consider the case where (3.4) is fulfilled. Since $\bar{w} \in W \subset R^n \times L$ and $\bar{w} + t_{k_1} z \in Z_2 \subset R^n \times L$, it follows that

$$\{\bar{w} + tz : t \in [0, t_{k_1}]\} \subset R^n \times L.$$

Therefore, we can deduce from the first equality in (3.1) that, for every $t \in (0, t_{k_1})$, $\bar{w} + tz \in Z_2$ or $\bar{w} + tz \in W$. If there exists $i \in N$ such that

$$\bar{w} + tz \in Z_2 \quad (\forall t \in (0, t_{k_i})) \tag{3.5}$$

then (c2) is satisfied if we choose $\bar{t} = t_{k_i}$. If there is no $i \in N$ such that (3.5) is valid, then for every $i \in N$ there must exist some $t'_{k_i} \in (0, t_{k_i})$ such that $\bar{w} + t'_{k_i} z \in W$. By (2.6), there is an index $j(k_i) \in \{1, \ldots, s\}$ such that

$$\bar{w} + t'_{k_i} z \in W_{j(k_j)}. \tag{3.6}$$

Without loss of generality we can assume that

$$0 < t'_{k_{i+1}} < t_{k_{i+1}} < t'_{k_i} < t_{k_i} \quad (\forall i \in N). \tag{3.7}$$

Since $j(k_i) \in \{1, \ldots, s\}$, there must exist a pair (i,j) such that $j > i$ and $j(k_j) = j(k_i)$. By (3.6) and by the convexity of $W_{j(k_i)}$, we have

$$\{\bar{w} + tz : t'_{k_j} \leqslant t \leqslant t'_{k_i}\} \subset W_{j(k_i)} \subset W. \tag{3.8}$$

From (3.4) and (3.7) we get $\varphi(\bar{w} + t_{k_{i+1}} z) = -\infty$ and $t'_{k_j} < t_{k_{i+1}} < t'_{k_i}$, a contradiction to (3.8). We have thus proved that if (3.4) is valid then (c2) must occur.

Summarizing all the above, we conclude that one of the three cases (c1)–(c3) must occur.

If (c1) occurs then by (3.2) we have $\varphi'(\bar{w}; z) = +\infty$. Similarly, if (c2) happens then $\varphi'(\bar{w}; z) = -\infty$. Now assume that (c3) takes place. Denote by F the collection of the index sets $I \subset \{1, 2, \ldots, m\}$ for which there exists $t_I \in (0, \bar{t})$, where $\bar{t} > 0$ is given by (c3), such that

$$\{\bar{w} + tz : t \in [0, t_I]\} \subset \operatorname{dom} \varphi^I_{\mathrm{KKT}}. \tag{3.9}$$

Observe that $\operatorname{dom} \varphi^I_{\mathrm{KKT}}$ is a closed convex set (see Lemma 2.4). If $F = \emptyset$ then for any $I \subset \{1, 2, \ldots, m\}$ and for any $t \in (0, \bar{t}]$ one has $\varphi^I_{\mathrm{KKT}}(\bar{w} + tz) = +\infty$. By (c3), $\bar{w} + tz \in W$ for all $t \in (0, \bar{t}]$. Then, according to (3.1) we have

$$\varphi(\bar{w} + tz) = \min\{\varphi^I_{\mathrm{KKT}}(\bar{w} + tz) : I \subset \{1, 2, \ldots, m\}\} = +\infty$$

for all $t \in (0, \bar{t}]$, which is impossible. We have shown that $F \neq \emptyset$. Define

$$\hat{t} = \min\{t_I : I \in F\} > 0.$$

By (c3) and (3.1) we have

$$\varphi(\bar{w} + tz) = \min\{\varphi^I_{\mathrm{KKT}}(\bar{w} + tz) : I \in F\} \quad (\forall t \in [0, \hat{t}\,]). \tag{3.10}$$

It follows from (3.9) that

$$\bar{w} + tz \in \operatorname{dom} \varphi^I_{\mathrm{KKT}} \quad (\forall I \in F, \forall t \in [0, \hat{t}\,]).$$

For each $I \in F$, let $M_I \in R^{(n+m)\times(n+m)}$ and $q_I \in R^{n+m}$ be such that the representation for $\varphi^I_{\mathrm{KKT}}(c, b)$ in Lemma 2.4 holds for all $(c, b) \in \operatorname{dom} \varphi^I_{\mathrm{KKT}}$. Setting

$$\tilde{\varphi}^I_{\mathrm{KKT}}(c, b) = \tfrac{1}{2}\begin{pmatrix} c \\ b \end{pmatrix}^T M_I \begin{pmatrix} c \\ b \end{pmatrix} + q_I^T \begin{pmatrix} c \\ b \end{pmatrix} \tag{3.11}$$

for every $(c,b) \in R^n \times R^m$, we extend $\varphi^I_{\mathrm{KKT}}(\cdot)$ from dom φ^I_{KKT} to the whole space $R^n \times R^m$. From (3.11) it follows that all the functions $\tilde{\varphi}^I_{\mathrm{KKT}}(\cdot)$, $I \in F$, are smooth. According to Clarke ([5], Theorem 2.1), the function

$$\tilde{\varphi}(c,b) = \min\{\tilde{\varphi}^I_{\mathrm{KKT}}(c,b) : I \in F\}$$

is locally Lipschitz at $\bar{w} = (\bar{c},\bar{b})$. Moreover, $\tilde{\varphi}$ is *Lipschitz regular* (see [6], Definition 2.3.4) at $\bar{w}$, and

$$\tilde{\varphi}^0(\bar{w};z) = \tilde{\varphi}'(\bar{w};z) = \min\{(\tilde{\varphi}^I_{\mathrm{KKT}})'(\bar{w};z) : I \in F\}, \tag{3.12}$$

where $\tilde{\varphi}^0(\bar{w};z)$ (resp., $\tilde{\varphi}'(\bar{w};z)$) denotes the Clarke generalized directional derivative (resp., the directional derivative) of $\tilde{\varphi}$ at $\bar{w}$ in direction z. Since $\tilde{\varphi}^I_{\mathrm{KKT}}(c,b) = \varphi^I_{\mathrm{KKT}}(c,b)$ for all $(c,b) \in$ dom φ^I_{KKT}, from (3.10) and (3.12) it follows that the directional derivative $\varphi'(\bar{w};z)$ exists, and we have

$$\varphi'(\bar{w};z) = \min\{(\varphi^I_{\mathrm{KKT}})'(\bar{w};z) : I \in F\}. \tag{3.13}$$

The proof is complete. □

In the course of the above proof we have obtained some explicit formulae computing the directional derivative of the function φ. Namely, we have proved the following result.

THEOREM 3.2. *Let $\bar{w} \in W$ and $z = (u,v) \in R^n \times R^m$. The following assertions hold:*

(i) *If there exists $\bar{t} > 0$ such that*
$\bar{w} + tz \in Z_1 = \{(c,b) : \Delta(A,b) = \emptyset\} \quad (\forall t \in (0,\bar{t}\,]),$
then $\varphi'(\bar{w};z) = +\infty$.

(ii) *If there exists $\bar{t} > 0$ such that*
$\bar{w} + tz \in Z_2 = \{(c,b) : \Delta(A,b) \neq \emptyset,\ \varphi(c,b) = -\infty\} \quad (\forall t \in (0,\bar{t}\,]),$
then $\varphi'(\bar{w};z) = -\infty$.

(iii) *If there exists $\bar{t} > 0$ such that*
$\bar{w} + tz \in W = \{(c,b) : \Delta(A,b) \neq \emptyset,\ \varphi(c,b) > -\infty\} \quad (\forall t \in (0,\bar{t}\,]),$
then $\varphi'(\bar{w};z)$ can be computed by formula (3.13), where F is the collection of the index sets $I \subset \{1,2,\ldots,m\}$ such that there exists some $t_I \in (0,\bar{t})$ satisfying condition (3.9).

At the end of Section 4 we shall use Theorem 3.2 for computing directional derivative of the optimal value function in a nonconvex QP problem.

4. The Piecewise Linear-quadratic Property

In this section we study the piecewise linear–quadratic property of the function $\varphi(\cdot)$ defined by (2.4).

DEFINITION 4.1 (see [11], p. 440). A function $\psi : R^l \to \bar{R}$ is piecewise linear–quadratic (plq) if the set

$$\text{dom}\ \psi = \{z \in R^l : -\infty < \psi(z) < +\infty\} \tag{4.1}$$

can be represented as the union of finitely many polyhedral convex sets, relative to each of which $\psi(z)$ is given by an expression of the form

$$\tfrac{1}{2}z^T Qz + d^T z + \alpha \tag{4.2}$$

for some $\alpha \in R,\ d \in R^l,\ Q \in R_S^{l\times l}$.

Note that in ([11], p. 440) instead of (4.1) one has the following formula:

$$\text{dom}\ \psi = \{z \in R^l : \psi(z) < +\infty\}. \tag{4.3}$$

If there exists some $\bar{z} \in R^l$ with $\psi(\bar{z}) = -\infty$ then, since $\bar{z}$ belongs to the set defined in (4.3), one cannot represent the latter as the union of finitely many polyhedral convex sets, relative to each of which $\psi(z)$ is given by an expression of the form (4.2). Hence ψ cannot be a plq function. This is the reason why we prefer (4.1)–(4.3).

In [11] a series of results on plq functions have been established. For example, it is proved that the conjugate of a proper, lower semicontinuous, convex function is plq if and only if the given function is plq (Theorem 11.14). It is shown that any proper, convex, plq, bounded below function has a global minimum (Corollary 11.16).

If D is a positive semidefinite matrix then, by using a theorem of Eaves (see [8], p. 825), one can prove that W is a polyhedral convex cone. Using Lemmas 2.4 and 2.5, and Remark 2.3, it is not difficult to show that *the optimal value function $\varphi(c, b) = \varphi(D, A, c, b)$ of a convex QP problem is plq*. This result is already known.

One anonymous referee of [14] informed that *the plq property of the optimal value function in convex QP problems was established by Sun* [12]. Being not able to consult the thesis of Sun, we have to limit our citation to Bank et al [2] and Rockafellar and Wets [11]. The referee asked: *Whether the optimal value function in a general (not necessarily convex) QP problem is a plq function w.r.t. its linear variables?* It turns out that the plq property is not available in the general case. By constructing an example we will give a negative answer to the question.

EXAMPLE 4.1. Consider the problem

$$\begin{aligned} &\text{minimize}\quad f(x,c) = \tfrac{1}{2}(x_1^2 + 2x_1x_2 - x_2^2) + c_1x_1 + c_2x_2,\\ &\text{subject to}\quad x = (x_1, x_2) \in R^2,\ \tfrac{1}{2}x_1 + x_2 \geqslant 0,\ x_2 - x_1 \geqslant 0,\ -x_2 \geqslant -2, \end{aligned} \tag{4.4}$$

and denote by $\varphi_1(c)$, $c = (c_1, c_2) \in R^2$, the optimal value of this nonconvex QP problem.

We will compute the values $\varphi_1(c)$, $c \in R^2$. Then it will be shown that $\varphi_1(c)$ is not a plq function. From the result it follows immediately that the optimal value function $\varphi(c, b)$, $c = (c_1, c_2) \in R^2$ and $b = (b_1, b_2, b_3) \in R^3$, of the following parametric QP problem is not plq:

$$\begin{aligned}
&\text{minimize } f(x,c) = \tfrac{1}{2}(x_1^2 + 2x_1x_2 - x_2^2) + c_1x_1 + c_2x_2 \\
&\text{subject to } x = (x_1,x_2) \in R^2,\ \tfrac{1}{2}x_1 + x_2 \geqslant b_1,\ x_2 - x_1 \geqslant b_2,\ -x_2 \geqslant b_3.
\end{aligned} \tag{4.5}$$

Let $\bar{b} = (0, 0, -2)$. In order to write (4.4) in the form (1.1), we put

$$D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad b = \bar{b} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Note that the feasible domain $\Delta(A, \bar{b})$ of (4.5) is a triangle with the vertices $(0, 0)$, $(2, 2)$ and $(-4, 2)$. Since $\Delta(A, \bar{b})$ is compact, $\varphi(c, \bar{b}) \in R$ for every $c \in R^2$. In other words, dom $\varphi(\cdot, \bar{b}) = R^2$. In agreement with (2.1) and (2.2), the auxiliary problem corresponding to (4.4) is the following one

$$\begin{aligned}
&\text{minimize } \quad \tfrac{1}{2}(c^Tx + b^T\lambda) = \tfrac{1}{2}(c_1x_1 + c_2x_2) - \lambda_3 \\
&\text{subject to } \quad (x, \lambda) = (x_1, x_2, \lambda_1, \lambda_2, \lambda_3) \in R^2 \times R^3, \\
&x_1 + x_2 - \tfrac{1}{2}\lambda_1 + \lambda_2 + c_1 = 0, \\
&x_1 - x_2 - \lambda_1 - \lambda_2 + \lambda_3 + c_2 = 0, \\
&\tfrac{1}{2}x_1 + x_2 \geqslant 0, \quad \lambda_1 \geqslant 0, \quad \lambda_1\left(\tfrac{1}{2}x_1 + x_2\right) = 0, \\
&x_2 - x_1 \geqslant 0, \quad \lambda_2 \geqslant 0, \quad \lambda_2(x_2 - x_1) = 0, \\
&x_2 \leqslant 2, \quad \lambda_3 \geqslant 0, \quad \lambda_3(2 - x_2) = 0.
\end{aligned} \tag{4.6}$$

We shall apply formula (2.10) to compute the values $\varphi(c, b)$, $c \in R^2$, $b = \bar{b}$. To do so, we have to compute the optimal value $\varphi^I_{\text{KKT}}(c, b)$ defined by (2.9), where $I \subset \{1, 2, 3\}$ is an arbitrary subset. For $I_1 := \{1, 2, 3\}$, taking account of (4.6), we obtain

$$\begin{aligned}
&\varphi^{I_1}_{\text{KKT}}(c, \bar{b}) = \tfrac{1}{4}(-c_1^2 - 2c_1c_2 + c_2^2), \\
&\text{dom } \varphi^{I_1}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : -3c_1 + c_2 \geqslant 0, \\
&c_2 \geqslant 0, \quad -c_1 + c_2 \leqslant 4\}.
\end{aligned} \tag{4.7}$$

The exact meaning of (4.7) is the following: $\varphi^{I_1}_{\text{KKT}}(c, \bar{b}) = \frac{1}{4}(-c_1^2 - 2c_1c_2 + c_2^2)$ for every c belonging to the above defined set dom $\varphi^{I_1}_{\text{KKT}}(\cdot, \bar{b})$ and

$$\text{dom } \varphi^{I_1}_{\text{KKT}}(c, \bar{b}) = +\infty$$

for every $c \notin \text{dom}\ \varphi^{I_1}_{\text{KKT}}(\cdot, \bar{b})$. A similar interpretation applies to the results of the forthcoming cases. For $I_2 := \{1, 2\}$, we have

$$\begin{aligned} &\varphi^{I_2}_{\text{KKT}}(c, \bar{b}) = -\tfrac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4, \\ &\text{dom}\ \varphi^{I_2}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : 2 - c_1 \geqslant 0, \\ &\qquad\qquad 4 + c_1 \geqslant 0,\ 4 + c_1 - c_2 \geqslant 0\}. \end{aligned} \tag{4.8}$$

For $I_3 := \{2, 3\}$, we have

$$\begin{aligned} &\varphi^{I_3}_{\text{KKT}}(c, \bar{b}) = 2c_1^2 + \tfrac{1}{2}c_2^2 - 2c_1c_2, \\ &\text{dom}\ \varphi^{I_3}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : c_2 \leqslant 3c_1, \\ &\qquad\qquad c_2 - 2c_1 \geqslant 0, \quad c_2 - 2c_1 \leqslant 2\}. \end{aligned} \tag{4.9}$$

For $I_4 := \{1, 3\}$, we have

$$\begin{aligned} &\varphi^{I_4}_{\text{KKT}}(c, \bar{b}) = -\tfrac{1}{4}(c_1 + c_2)^2, \\ &\text{dom}\ \varphi^{I_4}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : c_1 + c_2 \leqslant 0, \\ &\qquad\qquad c_1 + c_2 \geqslant -4, \quad c_2 \geqslant 0\}. \end{aligned} \tag{4.10}$$

For $I_5 := \{1\}$, we have

$$\begin{aligned} &\varphi^{I_5}_{\text{KKT}}(c, \bar{b}) = 2c_1 + 2c_2 + 4, \\ &\text{dom}\ \varphi^{I_4}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : c_1 + 4 \leqslant 0, \quad c_1 + c_2 + 4 \leqslant 0\}. \end{aligned} \tag{4.11}$$

For $I_6 := \{2\}$, we have

$$\begin{aligned} &\varphi^{I_6}_{\text{KKT}}(c, \bar{b}) = -4c_1 + 2c_2 - 2, \\ &\text{dom}\ \varphi^{I_6}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : c_1 - 2 \geqslant 0, \\ &\qquad\qquad 2 + 2c_1 - c_2 \geqslant 0\}. \end{aligned} \tag{4.12}$$

For $I_7 := \{3\}$, we have

$$\begin{aligned} &\varphi^{I_7}_{\text{KKT}}(c, \bar{b}) = 0, \\ &\text{dom}\ \varphi^{I_7}_{\text{KKT}}(\cdot, \bar{b}) = \{c = (c_1, c_2) \in R^2 : c_1 + c_2 \geqslant 0, \quad c_2 - 2c_1 \geqslant 0\}. \end{aligned} \tag{4.13}$$

For $I_8 := \emptyset$, we have

$$\begin{aligned} &\varphi^{I_8}_{\text{KKT}}(c, \bar{b}) = +\infty \quad \text{forevery } c \in R^2, \\ &\text{dom}\ \varphi^{I_8}_{\text{KKT}}(\cdot, \bar{b}) = \emptyset. \end{aligned} \tag{4.14}$$

Consider the following polyhedral convex subsets of R^2:

$$\begin{aligned}
\Omega_1 &= \{c = (c_1, c_2) : c_2 \leqslant -c_1 - 4, \quad c_1 \leqslant -4\},\\
\Omega_2 &= \{c = (c_1, c_2) : c_2 \geqslant -c_1 - 4, \quad c_2 \leqslant -c_1, \quad c_2 \geqslant c_1 + 4\},\\
\Omega_3 &= \{c = (c_1, c_2) : c_2 \geqslant -c_1, \quad c_2 \geqslant c_1 + 4, \quad c_2 \geqslant 2c_1 + 2\},\\
\Omega_4 &= \{c = (c_1, c_2) : c_2 \leqslant 2c_1 + 2, \quad c_2 \geqslant 2c_1 + 1, \quad c_1 \geqslant 2\},\\
\Omega_5 &= \{c = (c_1, c_2) : c_2 \leqslant 2c_1 + 1, \quad c_1 \geqslant 2\},\\
\Omega_6 &= \{c = (c_1, c_2) : c_1 \leqslant 2, \quad c_2 \leqslant 2c_1, \quad c_2 \geqslant 0\},\\
\Omega_7 &= \{c = (c_1, c_2) : c_2 \leqslant 0, \quad c_1 \geqslant -4, \quad c_1 \leqslant 2\},\\
\Omega_8 &= \{c = (c_1, c_2) : c_2 \geqslant 0, \quad c_2 \leqslant -c_1, \quad c_2 \leqslant (\sqrt{2} - 1)(c_1 + 4)\},\\
\Omega_9 &= \{c = (c_1, c_2) : c_2 \geqslant (\sqrt{2} - 1)(c_1 + 4), \quad c_2 \leqslant -c_1, \quad c_2 \leqslant c_1 + 4\},\\
\Omega_{10} &= \{c = (c_1, c_2) : c_2 \geqslant -c_1, \quad c_2 \leqslant c_1 + 4, \quad c_1 \leqslant 2, \quad c_2 \geqslant 2c_1\}.
\end{aligned}$$

Using formulae (2.10) and (4.7)–(4.14), we can show that

$$\begin{aligned}
&\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_5}(c, \bar{b}) = 2c_1 + 2c_2 + 4 \quad \text{for every } c \in \Omega_1,\\
&\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_4}(c, \bar{b}) = -\tfrac{1}{4}(c_1 + c_2)^2 \quad \text{for every } c \in \Omega_2 \cup \Omega_9,\\
&\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_7}(c, \bar{b}) = 0 \quad \text{for every } c \in \Omega_3 \cup \Omega_4,\\
&\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_6}(c, \bar{b}) = -4c_1 + 2c_2 - 2 \quad \text{for every } c \in \Omega_5,\\
&\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_2}(c, \bar{b}) = -\tfrac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4 \quad \text{for every } c \in \Omega_6 \cup \Omega_7 \cup \Omega_8.
\end{aligned}$$

We will pay a special attention to the behavior of $\varphi(\cdot, \bar{b})$ on the region Ω_{10}. Consider the parabola

$$\Gamma = \{(c_1, c_2) \in R^2 : c_2 = \tfrac{1}{4}c_1^2 + c_1 + 2\}.$$

It can be verified that, for each $c = (c_1, c_2) \in \Omega_{10}$,

$$\varphi(c, \bar{b}) = \begin{cases} 0 & \text{if } \ c_2 \geqslant \frac{1}{4}c_1^2 + c_1 + 2, \\ -\frac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4 & \text{if } \ c_2 \leqslant \frac{1}{4}c_1^2 + c_1 + 2. \end{cases} \tag{4.15}$$

Thus $\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_7}(c, \bar{b})$ for all the points $c \in \Omega_{10}$ lying above Γ, and $\varphi(c, \bar{b}) = \varphi_{\mathrm{KKT}}^{I_2}(c, \bar{b})$ for all the points $c \in \Omega_{10}$ lying below Γ.

PROPOSITION 4.1. *The obtained optimal value function $\varphi(c, \bar{b})$ $(c \in R^2)$ cannot be a piecewise linear–quadratic function.*

Proof. Suppose, contrary to our claim, that the function $\varphi(\cdot, \bar{b})$ is plq. Then the set dom $\varphi(\cdot, \bar{b}) = R^2$ can be represented in the form

$$R^2 = \bigcup_{j \in J} \Delta_j, \tag{4.16}$$

where J is a finite index set and Δ_j $(j \in J)$ are polyhedral convex sets. Moreover, for every $j \in J$, one has

$$\varphi(c, \bar{b}) = \tfrac{1}{2} c^T Q_j c + d_j^T c + \alpha_j \tag{4.17}$$

for all $c \in \Delta_j$, where $\alpha_j \in R,\ d_j \in R^2,\ Q_j \in R_S^{2\times 2}$. Let

$$\Delta'_j = \Delta_j \cap \Omega_{10} \quad (j \in J).$$

Note that some of the sets Δ'_j can be empty. From (4.16) we deduce that

$$\Omega_{10} = \bigcup_{j \in J} \Delta'_j. \tag{4.18}$$

Note also that on each set Δ'_j $(j \in J)$ the function $\varphi(\cdot, \bar{b})$ has the linear-quadratic representation (4.17). Define

$$\Omega_{10}^I = \{c = (c_1, c_2) \in \Omega_{10} : c_2 \geqslant \tfrac{1}{4} c_1^2 + c_1 + 2\},$$
$$\Omega_{10}^{II} = \{c = (c_1, c_2) \in \Omega_{10} : c_2 \leqslant \tfrac{1}{4} c_1^2 + c_1 + 2\}.$$

It is evident that Ω_{10}^I is a convex set. Note that Ω_{10}^I and Ω_{10}^{II} are compact sets which admit the curve $\Gamma \cap \Omega_{10}$ as the common boundary. The set Ω_{10}^I (resp., Ω_{10}^{II}) has nonempty interior. Indeed, let $\hat{c} := (0, 3)$ and $\tilde{c} := (0, 1)$. Substituting the coordinates of these vectors into the inequalities defining Ω_{10}^I and Ω_{10}^{II}, we see at once that $\hat{c} \in \operatorname{int} \Omega_{10}^I$ and $\tilde{c} \in \operatorname{int} \Omega_{10}^{II}$. Here and in the sequel, int M denotes the interior of a set M.

Fix any index $j \in J$ for which $\Delta'_j \neq \emptyset$. First consider the case int $\Delta'_j \neq \emptyset$. If

$$\operatorname{int} \Delta'_j \cap \operatorname{int} \Omega_{10}^I \neq \emptyset \tag{4.19}$$

then we must have $\Delta'_j \subset \Omega_{10}^I$. Indeed, by (4.19) there must exist a ball $B \subset R^2$ of positive radius such that $B \subset \Delta'_j \cap \Omega_{10}^I$. By (4.15), $\varphi(c, \bar{b}) = 0$ for every $c \in \Omega_{10}^I$. Then, it follows from (4.17) that

$$\varphi(c, \bar{b}) = \tfrac{1}{2} c^T Q_j c + d_j^T c + \alpha_j = 0$$

for every $c \in B$. This implies that $Q_j = 0,\ d_j = 0$ and $\alpha_j = 0$. Consequently,

$$\varphi(c, \bar{b}) = 0 \quad (\forall c \in \Delta'_j). \tag{4.20}$$

We observe from (4.15) that $\varphi(c, \bar{b}) < 0$ for every $c \in \Omega_{10} \setminus \Omega_{10}^I$. Hence (4.20) clearly forces $\Delta'_j \subset \Omega_{10}^I$. If $\operatorname{int} \Delta'_j \cap \operatorname{int} \Omega_{10}^I = \emptyset$ then we must have int $\Delta'_j \subset \Omega_{10}^{II}$. Since Ω_{10}^{II} is closed, we conclude that $\Delta'_j \subset \Omega_{10}^{II}$. Therefore, if $\Delta'_j \cap \Omega_{10}^I \neq \emptyset$ then $\Delta'_j \cap \Omega_{10}^I = \Delta'_j \cap \Gamma$. In this case, it is easy to show that $\Delta'_j \cap \Gamma$ is a singleton.

Now consider the case int $\Delta'_j = \emptyset$. Since Δ'_j is a compact polyhedral convex set in R^2, there are only two possibilities: i) Δ'_j is a singleton, and ii) Δ'_j

is a line segment. In both situations, if $\Delta'_j \cap \Omega^I_{10}$ is nonempty then it is a compact polyhedral convex set (a point or a line segment).

From (4.18) and from the above discussion, we can conclude that Ω^I_{10} is the union of the following finite collection of polyhedral convex sets:

$$\begin{aligned}
&\Delta'_j \quad (j \in J \text{ is such that} \quad \operatorname{int}\Delta'_j \cap \operatorname{int}\Omega^I_{10} \neq \emptyset),\\
&\Delta'_j \cap \Gamma \quad (j \in J \text{ is such that} \quad \operatorname{int}\Delta'_j \neq \emptyset,\ \operatorname{int}\Delta'_j \cap \operatorname{int}\Omega^I_{10} = \emptyset),\\
&\Delta'_j \cap \Omega^I_{10} \quad (j \in J \text{ is such that} \quad \operatorname{int}\Delta'_j = \emptyset,\ \Delta'_j \cap \Omega^I_{10} \neq \emptyset).
\end{aligned}$$

As Ω^I_{10} is convex, it coincides with the convex hull of the above-named compact polyhedral convex sets. According to Rockafellar ([9], Theorem 19.1), this convex hull is a compact polyhedral convex set. So it has only a finite number of extreme points (see [9], p. 162). Meanwhile, it is a simple matter to show that *every point from the infinite set* $\Gamma \cap \Omega_{10}$ *is an extreme point of* Ω^I_{10}. We have arrived at a contradiction. The proof is complete. $\square$

In Section 3 we have established a result on directional differentiability of the optimal value function $\varphi(c, b)$ of (1.1). Now we shall apply formula (3.13) to compute directional derivative of the function $\varphi(\cdot, \bar{b})$ studied in this section.

Let $\bar{c} = \bar{c}(\mu) = (0, \mu)$, $\mu \in R$. For $\bar{w}(\mu) = (\bar{c}(\mu), \bar{b})$ and $\bar{z} = (\bar{u}, \bar{v})$, where $\bar{u} = (1, 0) \in R^2$ and $\bar{v} = (0, 0, 0) \in R^3$, we have $\varphi'(\bar{w}(\mu); \bar{z}) = \varphi'_1(\bar{c}(\mu); \bar{u})$. Using formulae (3.13) and (4.7)–(4.14), we obtain

$$\varphi'(\bar{w}(\mu); \bar{z}) = \varphi'_1(\bar{c}(\mu); \bar{u})$$
$$= \begin{cases} (\varphi^{I_7}_{\mathrm{KKT}})'(\bar{c}(\mu); \bar{u}) & \text{for } \mu > 2,\\ \min\{(\varphi^{I_7}_{\mathrm{KKT}})'(\bar{c}(\mu); \bar{u}), & \\ (\varphi^{I_2}_{\mathrm{KKT}})'(\bar{c}(\mu); \bar{u})\} & \text{for } \mu = 2,\\ (\varphi^{I_2}_{\mathrm{KKT}})'(\bar{c}(\mu); \bar{u}) & \text{for } \mu < 2. \end{cases}$$

Therefore

$$\varphi'(\bar{w}(\mu); \bar{z}) = \varphi'_1(\bar{c}(\mu); \bar{u}) = \begin{cases} 0 & \text{for } \mu > 2,\\ -2 & \text{for } \mu \leqslant 2. \end{cases}$$

By Lemma 2.3, the function $\varphi_1(\cdot) = \varphi(\cdot, \bar{b})$ is locally Lipschitz on R^2. From Theorems 3.1 and 3.2 it follows that $\varphi_1(\cdot)$ is directionally differentiable at every $c \in R^2$ and, for every $u \in R^2$, the directional derivative $\varphi_1(c; u)$ is finite. One can expect that $\varphi_1(\cdot)$ is regular in the sense of Clarke [6], i.e., for every $c \in R^2$ it holds $\varphi^0_1(c; u) = \varphi'_1(c; u)$, where

$$\varphi^0_1(c; u) := \limsup_{c' \to c,\ t \downarrow 0} \frac{\varphi_1(c' + tu) - \varphi_1(c')}{t}$$

denotes the generalized directional derivative of φ_1 at c in direction u. Unfortunately, *the function* $\varphi_1(\cdot)$ *is not Lipschitz regular*. Indeed, for $\bar{c} = (0, 2)$ and $\bar{u} = (0, 1)$, using (4.15) it is not difficult to show that

$$0 = \varphi_1^0(\bar{c}; \bar{u}) > \varphi_1'(\bar{c}; \bar{u}) = -2.$$

5. Concluding Remarks

In this paper we have studied a class of optimal value functions in parametric (nonconvex) quadratic programming. It has been shown that these functions are directionally differentiable at any point from their effective domains but, in general, they are not piecewise linear–quadratic and they may be not Lipschitz regular at some interior points in their effective domains.

It would be desirable to find out what additional conditions one has to impose on the pair of matrices $(D, A) \in R_S^{n\times n} \times R^{m\times n}$, where D need not be a positive semidefinite matrix, so that the optimal value function

$$(c, b) \mapsto \varphi(D, A, c, b)$$

of the parametric problem (1.1) is piecewise linear–quadratic on $R^n \times R^m$.

Both referees of this paper informed us that Klatte had constructed an example of an optimal value function in a linearly perturbed QP problem which is not plq. Being unaware of that (unpublished) example, we have constructed Example 4.1. One referee gave us some hints in detail on the example of Klatte. Namely, letting two components of the data perturbation of a QP problem considered by Klatte [8] be fixed, one has the problem

$$\text{minimize} \quad x_1x_2 \quad \text{subject to} \quad x = (x_1, x_2) \in R^2,$$
$$-1 \leqslant x_1 \leqslant b_1, \quad b_2 \leqslant x_2 \leqslant 1,$$

where $b = (b_1, b_2) \in R^2$, $b_1 \geqslant 0$ and $b_2 \leqslant 0$, represents the perturbation of the feasible region. Denote by $\varphi(b_1, b_2)$ the optimal value function of this problem. It is easy to verify that

$$\varphi(b_1, b_2) = \begin{cases} -1 & \text{if } -1 \leqslant b_1b_2, \\ b_1b_2 & \text{if } -1 > b_1b_2. \end{cases}$$

If $b_1 < 0$ or $b_2 > 0$, then we put $\varphi(b_1, b_2) = +\infty$. Arguments similar to those of the proof of Proposition 4.1 show that $\varphi(b_1, b_2)$ is not a plq function. The main difference between this example and Example 4.1 is that here the feasible region is perturbed, while in Example 4.1 the objective function is perturbed. Note also that $\varphi_1(c)$ is a locally Lipschitz function defined on the whole space R^2.

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References

1. Auslender, A. and Coutat, P. (1996), Sensitivity analysis for generalized linear-quadratic problems, *Journal of Optimization Theory and Applications* 88, 541–559.
2. Bank, B., Guddat, J., Klatte, D., Kummer, B. and Tammer, K. (1982), *Non-Linear Parametric Optimization*, Akademie-Verlag, Berlin.
3. Best, M.J. and Chakravarti, N. (1990), Stability of linearly constrained convex quadratic programs, *Journal of Optimizaiton Theory and Applications* 64, 43–53.
4. Blum, E. and Oettli, W. (1972), Direct proof of the existence theorem for quadratic programming. *Operations Research* 20, 165–167.
5. Clarke, F.H. (1975), Generalized gradients and applications, *Transactions of the American Mathematics Society* 205, 247–262.
6. Clarke, F.H. (1983), *Optimization and Nonsmooth Analysis,* John Wiley & Sons, New York.
7. Cottle, R.W., Pang, J.-S. and Stone, R.E. (1992), *The Linear Complementarity Problem,* Academic Press, New York.
8. Klatte, D. (1985), On the Lipschitz behavior of optimal solutions in parametric problems of quadratic optimization and linear complementarity, *Optimization* 16, 819–831.
9. Rockafellar, R.T. (1970), *Convex Analysis,* Princeton University Press, Princeton, New Jersey.
10. Rockafellar, R.T. (1988), First- and second-order epi-differentiability in nonlinear programming, *Transactions of the American Mathematics Society* 307, 75–108.
11. Rockafellar, R.T. and Wets, R.J.-B. (1998), *Variational Analysis,* Springer, Berlin-Heidelberg.
12. Sun, J. (1986), Ph.D. Thesis, University of Washington.
13. Tam, N.N. (2001), Directional differentiability of the optimal value function in indefinite quadratic programming, *Acta Mathematica Vietnamica* 26, 377–394.
14. Tam, N.N. (2002), Continuity of the optimal value function in indefinite quadratic programming, *Journal of Global Optimizaiton* 23, 43–61.